Let B be a collection of subsets of a set X. Then B^* is a topology on X iff

(i) $\cup\{D : D \in B\} = X$ and

(ii) For each $D_1, D_2 \in B$ and each $x \in D_1 \cap D_2$ there exists $D \in B$ such that $x \in D \subseteq D_1 \cap D_2$

Proof – Let B^* be a topology on X. Then $X \in B^*$. Hence X is a union of members from B.

Therefore, $\cup\{D : D \in B\} = X$

Thus (i) holds.

Again since $B \subseteq B^*$, the members of B are open sets. Hence if $D_1, D_2 \in B$ then D_1, D_2 are open sets so $D_1 \cap D_2$ is an open set, hence $D_1 \cap D_2 \in B^*$ and thus $D_1 \cap D_2$ is a union of members from B.

Hence if $x \in D_1 \cap D_2$ there exists a $D \in B$ such that $x \in D \subseteq D_1 \cap D_2$

Conversely, suppose that the conditions (i) and (ii) hold. We shall show that B^* is a topology on X.

Clearly $\phi \in B^*$ for ϕ can be regarded as the union of an empty family of members from B. By condition (i)

X is a union of all members of B

and hence

$$X \in B^*$$

Thus the condition $[O_1]$ for a topology on X

is satisfied by B^*.

Again since each member of B^* is a union of members from B, an arbitrary union of members from B^* is a union of members from B and hence it belongs to B^*. Thus condition $[O_2]$ is also satisfied. Finally, let $G_1, G_2 \in B^*$

Then each of G_1, G_2 is a union of members from B. Hence if $x \in G_1 \cap G_2$, then $x \in G_1$, $x \in G_2$ and so there exists $D_1, D_2 \in B$ such that

$$x \in D_1 \subseteq G_1 \text{ and } x \in D_2 \subseteq G_2$$

Now

$$x \in D_1 \cap D_2,$$

hence by condition (ii) there exists $D \in B$ such that

$$x \in D \subseteq D_1 \cap D_2 \subseteq G_1 \cap G_2$$

Hence $G_1 \cap G_2$ is a union of members from B.

Hence

$$G_1 \cap G_2 \in B^*$$

and thus the condition $[O_3]$ is also satisfied.

Therefore, B^* is a topology on X.